Summary of VectorAnalysis

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0 Introduction

What I want to do, is to give students an short overview of the vectoranalysis. There are no conditions given about continuity, differentiabiliy, directions of vectors, etc. Speaking about fluids, keep always in your the different behaviour between compressible and incompressible fluids. Most of the time, the imagination is an incompressible fluid, when there is spoken about a fluid, which can be wrong.
Idea is that the students can read these conditions in books. But it is nice if you have an idea of what you are calculating and that you know how to derive a formula yourself. And to know the connections between the left and right side of an equal sign in a certain formula. Get some feeling for it.

1 Complex Numbers

- A complex number $z = a + b\,j$.
- Real part $\mathrm{Re}(z) = a$, Imaginary part $\mathrm{Im}(z) = b$.
- Conjugate of z, notation: z^*, $z^* = a - b\,j$.
- $j^2 = -1$.
- Modulus of z, $|z| = \sqrt{a^2 + b^2}$.
- $|z|^2 = z\,z^*$.
- Argument of z, notation: $\arg(z)$, angle of z with the positive real axis, besides an integer multiple of 2π. The principal value of z, notation: $\mathrm{Arg}(z)$, $0 \leq \mathrm{Arg}(z) < 2\,\pi$.
- Formula of Euler: $e^{(\phi\, j)} = \cos(\phi) + j\,\sin(\phi)$.
- Another way of writing: $z = |z|\,e^{(\arg(z)\,j)}$.
- There are some basic rules to multiply or to divide complex numbers.

  Multiplication:
  $|z_1\,z_2| = |z_1|\,|z_2|$, $\arg(z_1\,z_2) = \arg(z_1) + \arg(z_2)$.

  Division:
  $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$, $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$.

2 Differential equations

The rules to calculate a solution for an ordinary second order differential equation with constant coefficients. The parameter is called t, in this section and the differential equations is taken equal to:

$$m\, \ddot{x}\,(t) + b\, \dot{x}\,(t) + k\, x(t) \;=\; y(t). \tag{2.0.1}$$

the constants m, b, and k are real numbers. Split the differential equation in a homogeneous equation and a inhomogeneous equation. First solve the homogeneous equation, giving the homogeneous solution $x_H(t)$, and secondly, surch for the solution of the inhomogeneous equation, giving the particular solution, $x_P(t)$. Filling the particular solution, into the differential equation it must give the inhomogeneous term, $y(t)$, of the differential equation.
The total solution becomes $x_T(t) \;=\; x_H(t) \;+\; x_P(t)$.
Last step: When there are given initial or boundary conditions, you have to calculate the unknown constants, psrt of the solution of the homogeneous equation.

2.1 Solutions of the homogeneous equation

- Solutions have always the form: $A\, \mathrm{e}^{\lambda\, t}$, A is a number.
- λ satisfies the equation $m\, \lambda^2 \;+\; b\, \lambda \;+\; k \;=\; 0$.
- The equation $m\, \lambda^2 \;+\; b\, \lambda \;+\; k \;=\; 0$ has always two solutions, λ_1 and λ_2. The type of these solutions:

  a) λ_1 and λ_2 are real and $\lambda_1 \;\neq\; \lambda_2$

  b) $\lambda_1 \;=\; \lambda_2$

  c) λ_1 and λ_2 are complex, so $\lambda_2 \;=\; \lambda_1^*$
- Case a) gives us the homogeneous solution:
  $x_H(t) \;=\; A\, \mathrm{e}^{(\lambda_1\, t)} \;+\; B\, \mathrm{e}^{(\lambda_2\, t)}$, A and B are numbers.
- Case b) gives us the homogeneous solution:
  $x_H(t) \;=\; A\, \mathrm{e}^{(\lambda_1\, t)} \;+\; B\, t\, \mathrm{e}^{(\lambda_1\, t)}$, A and B are numbers.
- Case c) gives us the homogeneous solution:
  if $\lambda_1 \;=\; a \;+\; b\, j$ then
  $x_H(t) \;=\; A\, \mathrm{e}^{(a\, t)}\, \cos(b\, t) + B\, \mathrm{e}^{(a\, t)}\, \sin(b\, t)$, A and B are real numbers.

- Case c) and calculating with complex numbers, gives us the homogeneous solution: $x_H(t) = A\,\mathrm{e}^{(\lambda_1 t)} + B\,\mathrm{e}^{(\lambda_2 t)}$, A and B are complex numbers.

With $Pol_i(t, n)$ is meant a polynomial in t with a degree $\leq$ n, so $Pol_i(t, n) = a_0\,t^0 + a_1\,t^1 + a_2\,t^2 + \ldots + a_n\,t^n$ with coëfficients a_i, which are real or complex, depending on the problem, i is given to distinguish the polynomials. Put the particular solution into the equation and calculate the not-known coëfficients. (Take equal terms together, get linear equations and solve them.)

2.2 Particular solution

- $y(t) = Pol_1(t, n)$, try

  ■ 0) Case a) and $\lambda_i \neq 0$ then $x_P(t) = Pol_2(t, n)$,

  ■ 1) Case a) and $\lambda_1 = 0$ and $\lambda_1 \neq 0$ then $x_P(t) = Pol_3(t, n+1)$, without the term t^0, or $x_P(t) = t\,Pol_2(t, n)$,

  ■ 2) Case b) and $\lambda_1 = 0$ then $x_P(t) = Pol_4(t, n+2)$, without the terms t^0 and t^1, or $x_P(t) = t^2\,Pol_2(t, n)$

- $y(t) = p_1 * \mathrm{e}^{(a\,t)} \cos(b\,t) + p_2 * \mathrm{e}^{(a\,t)} \sin(b\,t)$, try

  ■ 0) If $\lambda_i \neq a + b\,j$, then $x_P(t) = q_1 * \mathrm{e}^{(a\,t)} \cos(b\,t) + q_2 * \mathrm{e}^{(a\,t)} \sin(b\,t)$

  ■ 1) If $\lambda_i = a + b\,j$, then $x_P(t) = (q_1 + q_3\,t) * \mathrm{e}^{(a\,t)} \cos(b\,t) + (q_2 + q_4\,t) * \mathrm{e}^{(a\,t)} \sin(b\,t)$ (Very much work!)

- $y(t) = p_1 * \mathrm{e}^{(a\,t)} \cos(b\,t) + p_2 * \mathrm{e}^{(a\,t)} \sin(b\,t)$, most of the time it is easier to rewrite the inhomogeneous term in complex exponential functions.

  ■ 0) If $\lambda_i \neq a + b\,j$ and $p_2 = 0$ then $y(t) = \mathrm{Re}(p_1 * \mathrm{e}^{((a + b\,j)\,t)})$, try the complex particular solution $cx_P(t) = cq_1 * \mathrm{e}^{((a + b\,j)\,t)}$, cq_1 is a complex number, and after all the calculations $x_P(t) = \mathrm{Re}(cx_P(t))$.

  ■ 1) If $\lambda_i \neq a + b\,j$ and $p_1 = 0$ then $y(t) = \mathrm{Im}(p_2 * \mathrm{e}^{((a + b\,j)\,t)})$, try the complex particular solution $cx_P(t) = cq_2 * \mathrm{e}^{((a + b\,j)\,t)}$, cq_2 is a complex number, and after all the calculations $x_P(t) = \mathrm{Im}(cx_P(t))$.

- $y(t) = Pol_1(t, n) * \mathrm{e}^{((a + b\,j)\,t)}$, try

  ■ 0) If $\lambda_i \neq a + b\,j$, try $x_P(t) = Pol_2(t, n) * \mathrm{e}^{((a + b\,j)\,t)}$.

■ 1) If $\lambda_i = a + b\,j$, try $x_P(t) = Pol_3(t, n+1) * \mathrm{e}^{((a + b\,j)\,t)}$.

3 Inner product, outer product

There is worked in an orthonormal Cartesian counterclockwise rotating coördinate system. (Be careful with the zero-vector, ϕ is the angle between the two vectors.)

3.1 Inner product

Two vectors $\underline{v}$ and $\underline{w}$ then is the definition of inner product:
$\underline{v} \bullet \underline{w} = |\underline{v}|\,|\underline{w}| \cos\phi$, with $|-|$ the length of a vector and ϕ the angle between the two given vectors. In coördinates, if the two given vectors have the following coördinates

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{and} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

then the inner product becomes

$$\underline{v} \bullet \underline{w} = v_1\,w_1 + v_2\,w_2 + v_3\,w_3.$$

The inner product is a number. Properties:

- ♦ $\underline{v} \bullet \underline{w} = |\underline{v}|\,|\underline{w}| \cos\phi$.
- ♦ $\underline{v} \bullet \underline{w} = 0 \Longleftrightarrow \underline{v} \perp \underline{w}$.
- ♦ $\underline{v} \bullet \underline{w} = \underline{w} \bullet \underline{v}$.
- ♦ $\underline{u} \bullet (\underline{v} + \gamma\,\underline{w}) = \underline{u} \bullet \underline{v} + \gamma\,\underline{u} \bullet \underline{w}$, with γ a number.

3.2 Outer product

The outer product $\times$ between the vector $\underline{v}$ and $\underline{w}$ gives a vector as result. The vector $\underline{u} = \underline{v} \times \underline{w}$ is the outer product of $\underline{v}$ and $\underline{w}$.
Properties:

- $|\underline{u}| = |\underline{v} \times \underline{w}| = |\underline{v}|\,|\underline{w}| \sin\phi$, the area of the parallellogram made by the vectors $\underline{v}$ and $\underline{w}$.
- $\underline{u} \perp \underline{v}$, $\underline{u} \perp \underline{w}$.

- $\underline{v} \times \underline{w} = -\underline{w} \times \underline{v}$.
- $\underline{u} \times (\underline{v} + \gamma\, \underline{w}) = \underline{u} \times \underline{v} + \gamma\, \underline{u} \times \underline{w}$, with γ a number.

How to calculate the outer product:

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{pmatrix} (v_2\, w_3 - w_2\, v_3) \\ -(v_1\, w_3 - w_1\, v_3) \\ (v_1\, w_2 - w_1\, v_2) \end{pmatrix}$$

with $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ vectors with length 1 in the direction of the x, y and z-axis.

4 Lines and planes

4.1 Line

Given a position point $\underline{p}$ in the space and some direction vector $\underline{v}$. A parametrization of this line is

$$\text{line} := \underline{p} + \lambda\, \underline{v}.$$

There is one parameter: λ, because a line is one dimensional.

4.2 Plane

4.2.1 Parametrization

Given a position point $\underline{p}$ in the space and two direction vectors $\underline{v}$ and $\underline{w}$. A parametrization of this plane is

$$\text{plane} := \underline{p} + \lambda\, \underline{v} + \mu\, \underline{w}.$$

There are two parameters: λ and μ, because a plane is two dimensional.
This going well, if the direction vectors $\underline{v}$ and $\underline{w}$ are linear independent. They have not to be multiple values of each other. They don't have the same direction or the opposite direction of each other.

4.2.2 Linear equation

The direction vectors $\underline{v}$ and $\underline{w}$ are parallel to the plane and linear independent, so the normal of the plane is $\underline{n} = \underline{v} \times \underline{w}$.
The linear equation of the plane becomes $\underline{n} \bullet (\underline{x} - \underline{p}) = 0$, all the vectors $\underline{x}$ satisfying this equation are lying on the given plane.

4.3 Lines and linear equations

A line can be seen as the section of two planes. So, you need two linear equations to describe a line.

5 Linearization of a function

5.1 One variable

When you want to approximate some function value of the function $f(t)$, in the neighbourhood of t_0, knowing the values $f(t_0)$ and the first derivative $f'(t_0)$, you have to calculate $f(t_0 + \triangle t) \doteq f(t_0) + f'(t_0)\,\triangle t$. Here the function is linearized.
You can also calculate the difference between the function values $\triangle\, f(t_0) = f(t_0 + \triangle t) - f(t_0) \doteq f'(t_0)\,\triangle t$. Written with differentials ($\triangle t \rightarrow 0$) you have $df(t_0) = f'(t_0)\,dt$.

5.2 Several variables (3 variables)

A function f of three variales, the variables are the coordinates of the vector $\underline{x}$. Taking two variables fixed and one free and you have a function of one variable and you can linearize this function. Do this for all the variables and you get $f(\underline{x}_0 + \triangle\underline{x}) \doteq f(\underline{x}_0) + \text{grad}f(x_0) \bullet \triangle\underline{x}$, with the vector $\text{grad}f(\underline{x})$. The gradient of the function f, grad$f$, you made by calculating the partial derivatives of the function f and putting them in a vector. The gradient gives you the direction in which the function has its greatest increasement. Letting $\triangle\underline{x} \rightarrow 0$ you get $df(\underline{x}_0) = \text{grad}f(x_0) \bullet d\underline{x}$.

6 Parametrized curves and surfaces

6.1 Curves, tangent lines

$\mathcal{C}$ is a curve in the 3 dimensional space. There is a parametrization of this curve, this means that $\mathcal{C}$ can written as follows $\mathcal{C} := \underline{x}(t)$, t is the parameter. A line is one dimensional, so just 1 parameter. A tangent line to this curve can be calculated by differentiating the parametrization. Differention gives us the velocity vector $\underline{v}(t)$ at some point $\underline{x}(t)$ on the curve. A parametrization of such a tangent line TL at $t = t_0$ is $TL(\lambda) = \underline{x}(t_0) + \lambda\,\underline{v}(t_0) = \underline{x}(t_0) + \lambda\,\frac{\partial \underline{x}}{\partial t}(t_0)$.

6.2 Surfaces, tangent planes

$\mathcal{S}$ is a surface in the 3 dimensional space. There is a parametrization of this surface, this means that $\mathcal{S}$ can written as follows $\mathcal{S} := \underline{x}(s,t)$, s, t are the parameters. A surface is two dimensional, so just 2 parameters. A tangent plane to this surface can be calculated by differentiating the parametrization. If you keep one parameter constant and the otherone free, you get a curve in space and you can calculate tangent line of this curve at place $\underline{x}(s_0,t_0)$. Doing twice you get two direction vectors of the tangent plane $\frac{\partial \underline{x}}{\partial s}(s_0,t_0)$ and $\frac{\partial \underline{x}}{\partial t}(s_0,t_0)$ and take $\underline{x}(s_0,t_0)$ as the position vector. You can parametrize the tangent plane or you can calculate the linear equation of the tangent plane at some point of the surface.

6.2.1 Special case, the gradient

If the surface is given by the following equation $f(\underline{x}) = 0$, then the normal of a tangent plane, is easily calculated with the $\mathrm{grad}(f)$. The gradient gives us the direction of the greatest increasement of a function, so it stays perpendicular to the tangent plane.

7 Line integrals

Functions integrating over a curve are giving you line integrals.

7.1 Integral over parametrized curve

The curve is $\mathcal{C} := \underline{x}(t)$, with the parameter t, with $a \leq t \leq b$ and the function you want to integrate is f. Along the curve you have $dx = \underline{x}'(t)\,dt$ and this means $|dx| = |\underline{x}'(t)|\,dt$ and $|dx|$ is the distance along the curve, so you can write $ds = |dx| = |\underline{x}'(t)|\,dt$.
If you want to calculate the length of a curve, you have to integrate a function which gives you the value 1 everywhere on the curve and $\mathrm{L}(S) = \int_{\mathcal{C}} 1\,ds = \int_a^b |\underline{x}'(t)|\,dt$.
Integrating the function f over the given curve gives you $\mathrm{Int}_{\mathcal{C}}(f) = \int_{\mathcal{C}} f(\underline{x})\,ds = \int_a^b f(\underline{x}(t))\,|\underline{x}'(t)|\,dt$.
So $ds = |\underline{x}'(t)|\,dt$, and when you have some mass distribution ρ and youwant to calculate the mass of the curve $\mathrm{M}(\mathcal{C}) = \int_{\mathcal{C}} dm$, you get $dm = \rho(\underline{x})\,ds = \rho(\underline{x}(t))\,|\underline{x}'(t)|\,dt$. The function f has become the function ρ of the mass distribution.

7.2 Vectorfield integrating over a curve

With the vectorfield $\underline{F}$ and the term $d\underline{x}$, you make the function $\underline{F} \bullet d\underline{x}$ and this function you want to integrate over a curve. This gives you $\int_{\mathcal{C}} \underline{F} \bullet d\underline{x} = \int_a^b \underline{F}(\underline{x}(t)) \bullet \underline{x}'(t)\, dt$.
In certain sense you calculates the amount of work, which has te be done to become from a certain point in space to another one, over some curve in space.
Another notation for such a line integral, if $\underline{F} = (F_1, F_2, F_3)$,

$$\int_{\mathcal{C}} \underline{F} \bullet d\underline{x} = \int_{\mathcal{C}} (F_1\, dx + F_2\, dy + F_3\, dz),$$

you have just written out the inner product.

7.2.1 Special case, the potential

If the vectorfield $\underline{F}$ has a special form, it's meant that $\underline{F}$ can be written in the following form $\underline{F}(\underline{x}) = -\text{grad}(\phi)(\underline{x})$, then you get $\underline{F}(\underline{x}) \bullet d\underline{x} = -\text{grad}(\phi)(\underline{x}) \bullet d\underline{x} = -d\phi(\underline{x})$. See the linearization of a function with several variables. And this you can integrate giving $\int_{\mathcal{C}} \underline{F} \bullet d\underline{x} = \int_{\mathcal{C}} -d\phi = \phi(\underline{x}(a)) - \phi(\underline{x}(b))$. The function ϕ is called the potential of the vectorfield $\underline{F}$. You also see the integral doesn't depend on the path you are going from the begin point to the endpoint of the curve. But keep aware of singularities, running with the wind in your back is easier then the wind against you. Think to a tornado, with such an eye in it. These kind of phenomena can be described with vectorfields of the form just described The eye of a tornado is the singularity. The work to be done to go around that eye differs very much of the direction you take!
Be aware of the minus sign before the gradient of the potential.

8 Double integrals

This are integrals over two dimensional domains. To describe the domain, you need parametrizations of the boundaries of your domain. Let's call the domain G and the function f.
You want to calculate $\int\int_{\mathrm{G}} f\, d\mathcal{O}$, some integral of a function f over the domain G, with $d\mathcal{O}$, you have the freedom to choose your own coordinates. But most of the time you are integrating over a domain given in the Cartesian coordinates x and y, so you get
$d\mathcal{O} = dx\, dy$.

8.1 Cartesian coordinates

Most of the time you have integrals with the integration variables x and y and you want to describe the boundary of your domain in the variable x or y. Try to get for one of the integration variables fixed values and look how the boundary of your domain behaves.
For instance:
You have chosen to take $a \leq x \leq b$. Take some x, between a and b, and look what the boundaries are from the variable y. In this example $g(x) \leq y \leq h(x)$. Integrating the funcion f over this domain, you get $\int\int_G f(x,y)\,dx\,dy = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$. When you calculate, first integrate over the variable y and then over the variable x. So work from inside to the outside of the integral.
The same story, as above, you can read, when you take fixed values for the variable y, but change everywhere the names of x and y.

8.2 Polar coordinates

There are also the polar coordinates R, ϕ. With the complex numbers you have already used them,

$$\left[\begin{array}{lcl} x & = & R\cos(\phi) \\ y & = & R\sin(\phi) \end{array}\right.$$

with $R \geq 0$ and $0 \leq \phi < 2\pi$.
Be careful when you calculate ϕ, it is not simple $\arctan(\frac{y}{x})$!
Using the Cartesian coordinates x and y, you have cut the domain in little rectangles and you got $d\mathcal{O} = dx\,dy$. Looking in the xy-plane, you can divide the domain with polar coordinates in little pieces. You can divide the radius in pieces and also the angle ϕ. Taking $R + dR$ and $\phi + d\phi$ and looking at the point (R,ϕ) in polar coordinates, you get a rectangle and the area of this area is $R\,dr\,d\phi$. You get the following important relation

$$d\mathcal{O} = dx\,dy = R\,dr\,d\phi.$$

Changing from Cartesian to Polar coordinates you have to use this correction term.

8.3 Mass distribution

Do you know the mass distribution ρ on the domain of integration, then you get

$$dm = \rho\,d\mathcal{O} = \rho\,dx\,dy = \rho\,R\,dr\,d\phi.$$

9 Surface integrals

With such an integral you want to integrate a function f over a surface $\mathcal{S}$. For instance when you want to calculate the mass of a bubble. Idea behind it is that the surface $\mathcal{S}$ is two dimensional, so you can parametrize it with two variables. The integral you want to calculate has to be written in the form of a double integral. But when you project a two dimensional surface on a flat plane (for instance xy-plane), the area of a part of the surface becomes smaller, not always. (Think of two planes, which stay almost perpendicular to each other and project a little of one plane to the other plane and compare the areas of these pieces.) It is true, when you take $z = g(x, y)$ and project that surface on the xy-plane. To compensate this fact, you have to calculate a correction term. The function times the correction term becomes a function on certain domain and you can use the theory of double integrals. The parametrization of your surface $\mathcal{S}$ determines the domain of integration and the correction term.

9.1 The correction term, 2D

You want to calculate not a $d\mathcal{O}$ but a $d\mathcal{S}$ term, the area of a surface element. You know that $d\mathcal{S}$ with $\mathcal{S} := \underline{x}(s, t)$, s, t are the parameters, is equal to the length of the following outer product

$$(\frac{\partial \underline{x}(s,t)}{\partial t} dt) \times (\frac{\partial \underline{x}(s,t)}{\partial s} ds)$$

So you have found that

$$d\mathcal{S} = |\frac{\partial \underline{x}(s,t)}{\partial t} \times \frac{\partial \underline{x}(s,t)}{\partial s}|\, ds\, dt$$

If the parameters s and t are the cartesian coordinates x and y, you have

$$d\mathcal{S} = |\frac{\partial \underline{x}(x,y)}{\partial x} \times \frac{\partial \underline{x}(x,y)}{\partial y}| d\mathcal{O}.$$

When you are doing something with a mass distribution ρ you know

$$dm = \rho\, d\mathcal{S}.$$

9.1.1 Other correction terms in the xy-plane

You see that that the correction term of the polar coordinates can be derived on this way. See the xy-plane as a surface in the three dimensional space

$(x, y, 0)$, just take the z-coordinate equal to zero! Having other coordinates in the xy-plane, you can now calculate the belonging correction terms. You are calculating the area of the surface spanned by two direction vectors, coming from the partial derivatives of the parameterization.

9.2 The flux

You can make your own functions to calculate over a surface $\mathcal{S}$. For instance, looking at a flow of air or water with a velocity $\underline{v}$, you can ask yourself, how much of this flow is going through the surface $\mathcal{S}$. You mean the volume of the flow going through that surface, within a unit of time, also called the Flux($\mathcal{S}$) of the flow through the surface $\mathcal{S}$. If you want the mass, then you also need the mass density ρ.
The velocity of the flow is given by the vectorfield $\underline{v}$. Flow parallel to a tangent plane of that surface is not relevant, only the part of the flow perpendicular to a tangent plane. So you get that the amount of the flow dF going through a small part of the surface surface $d\mathcal{S}$ is given by

$$d\mathrm{F} \;=\; (\underline{\mathrm{v}} \bullet \underline{\mathrm{n}})\, d\mathcal{S}.$$

with $\underline{\mathrm{n}}$ the normal on the surface $\mathcal{S}$, this normal has length 1. The Flux($\mathcal{S}$) becomes

$$\mathrm{Flux}(\mathcal{S}) \;=\; \int\int_{\mathcal{S}} d\mathrm{F} \;=\; \int\int_{\mathcal{S}} \underline{\mathrm{v}} \bullet \underline{\mathrm{n}}\, d\mathcal{S}.$$

10 Volume integrals

Integrals of a function over a volume. When you know how to calculate double integrals, the volume integrals are going on almost the same way, there is only one variable more. You have to do with a volume element $d\mathcal{V}$ and not with $d\mathcal{S}$ or $d\mathcal{O}$. In some cases, you have a relation between an integral over a volume and an integral over the boundary of that particular volume.
An example, if you have a volume and in that volume is some production of mass. You can make an integral over the whole volume to see how much mass is produced, within a unit of time. Another way is that you calculate the amount of mass going through the boundary of that volume, within an unit of time. The first is a volume integral, the second is a surface integral. There has to be a relation between these integrals.
For instance you are asking yourself what to do, when you are using other variables as the Cartesian coordinates. With polar coordinates you saw such a correction term, using other variables you get also such a correction term,

but how to calculate?

10.1 Correction term, 3D

In the two dimensional case, you calculated the relation of a surface element to the parameters you have chosen. This is easily done by taking the partial derivatives of a parametrization, giving you tangent vectors on a surface and the length of the outer product of these tangent vectors gives you the correction term.
You have a volume $\mathcal{V}$, parametrized by $\mathcal{V} := \underline{x}(s,t,u)$, s, t and u are the parameters. The $d\mathcal{V}$ is something of a volume and volume is area times height. Taking one of the variables constant in the parametrization then you get a surface and you can calculate $d\mathcal{S}$ at some point in the volume $\mathcal{V}$. Question is how you can calculate the height of $d\mathcal{V}$, the distance perpendicular to $d\mathcal{S}$. If you have this height, you have $d\mathcal{V}$.
When the variable $u = u_0$ is taken constant and the variables s and t are free to choose, the following vectors

$$pds_{ds} = \frac{\partial \underline{x}(s_0, t_0, u_0)}{\partial s}\, ds \quad \text{and} \quad pdt_{dt} = \frac{\partial \underline{x}(s_0, t_0, u_0)}{\partial t}\, dt$$

describe a parallellogram to calculate the term $d\mathcal{S}$ at some point $\underline{x}(s_0, t_0, u_0)$. The outer product of these two vectors stays perpendicular to that parallellogram. Now you can calculate the following vector

$$pdu_{du} = \frac{\partial \underline{x}(s_0, t_0, u_0)}{\partial u}\, du$$

and take the inner product with

$$pds_{ds} \times pdt_{dt}.$$

Your result is

$$\begin{gathered}(pds_{ds} \times pdt_{dt}) \bullet pdu_{du} = \\ |\, pds_{ds} \times pdt_{dt}\,|\,|\, pdu_{du}\,| \cos(\text{angle between}\,(pds_{ds} \times pdt_{dt})\,\text{and}\, pdu_{du})\end{gathered}.$$

Look at your result, the last part of the expression you explain as the length of the projection of pdu_{du} at yhe vector $(pds_{ds} \times pdt_{dt})$, that is the height of $d\mathcal{V}$ you searched. The first term $|\, pds_{ds} \times pdt_{dt}\,|$ is the area of the parallellogram spanned by pds_{ds} and pdt_{dt}, the well-known $d\mathcal{S}$. You know that the

inner product can be negative, and volumes are always positive, so take the absolute value of the calculated inner product

$$d\mathcal{V} = |(\frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial s}) \bullet \frac{\partial \underline{x}}{\partial u}|\, ds\, dt\, du.$$

You have calculated the volume of a parallellopipedum spanned by the vectors pds_{ds}, pdt_{dt} and pdu_{du}.
You can find another way to calculate the volume of a parallellopipedum, and find something, which is called a determinant. It differs not so much, as what you are doing to the calculation of an outer product, only the result is a number and not a vector. Looking in books about calculus or analysis, you will also read the term Jacobian.

10.2 Mass

You know that the formula of the mass of a certain volume is

$$\mathrm{M}(\mathcal{V}) = \int\int\int_{\mathcal{V}} dm = \int\int\int_{\mathcal{V}} \rho\, d\mathcal{V},$$

when ρ it the mass density, because $dm = \rho\, d\mathcal{V}$.

11 The Divergence Theorem of Gauss

The first thing you ask yourself, what'd divergence? Also you had some earlier an idea that the flux has some something to do with a certain volume integral. May be the divergence is helping you?
In your volume $\mathcal{V}$, you investigate at some fixed point (x_0, y_0, z_0), a little box, with the measures $\triangle x$, $\triangle y$ and $\triangle z$, this are Cartesian coordinates.
You are looking at two sides of the box, the first side has the coordinates (x_0, y_0, z_0), $(x_0, y_0 + \triangle y, z_0)$, $(x_0, y_0, z_0 + \triangle z)$, $(x_0, y_0 + \triangle y, z_0 + \triangle z)$ and the second site is parallel to the first site but the x_0 is taken equal to $x_0 + \triangle x$.
The volume of the fluid coming into the site with the coordinate x_0 is equal to

$$\underline{v}(x_0, y_0, z_0)\, \triangle y\, \triangle z$$

and the volume of the fluid coming into the site with the coordinate $x_0 + \triangle x$ is equal to

$$\underline{v}(x_0 + \triangle x, y_0, z_0)\, \triangle y\, \triangle z.$$

Using a linear approximation, you get

$$\begin{aligned} &\underline{v}(x_0 + \triangle x, y_0, z_0)\, \triangle y\, \triangle z - \underline{v}(x_0, y_0, z_0)\, \triangle y\, \triangle z \\ &\doteq \frac{\partial \underline{v}}{\partial x}(x_0, y_0, z_0)\, \triangle x\, \triangle y\, \triangle z. \end{aligned}$$

You have calculated the change of volume in the x-direction per unit of time. The same can be done for the y- and z-direction. The total change of volume in the little box is

$$(\frac{\partial \underline{v}}{\partial x} + \frac{\partial \underline{v}}{\partial y} + \frac{\partial \underline{v}}{\partial z}) \triangle x \triangle y \triangle z = \mathrm{div}(\underline{v}) \triangle x \triangle y \triangle z.$$

You are looking at the velocity $\underline{v}$ and then the divergence, notation: $\mathrm{div}(\underline{v})$ gives you the fraction with which the volume of the fluid changes per unit of time, in certain sense the source strength. tThis means that integrating this fraction, per unit of time, over the whole volume gives you the total change of the volume per unit of time. That is equal to the flux, if you take for $\mathcal{S}$ the closed boundary of the volume $\mathcal{V}$.
The integral theorem of Gauss or the divergence theorem becomes

$$\int\int\int_{\mathcal{V}} \mathrm{div}(\underline{v})\, d\mathcal{V} = \int\int_{\mathcal{V}} (\underline{v} \bullet \underline{n})\, d\mathcal{S},$$

with

$$\mathrm{div}(\underline{v}) = \frac{\partial \underline{v}}{\partial x} + \frac{\partial \underline{v}}{\partial y} + \frac{\partial \underline{v}}{\partial z}$$

the divergence of the vectorfield $\underline{v}$ in this case.
Most of the time, there is another way of writing of the divergence, you can write it as an inner product

$$\mathrm{div}(\underline{v}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \underline{v} = \bigtriangledown \bullet \underline{v}$$

The operator $\bigtriangledown$, you can also see, as the operator needed to calculate the gradient of a function, $\mathrm{grad}(f) = \bigtriangledown(f)$.
You have changed a volume integral of a certain kind into a surface integral. The question comes in mind if you have also certain surface integrals, which can be changed into line integrals?

12 A fluid moving in the space, rotation

You can ask yourself, what's happening with a particle moving along some curve $\mathcal{C} := \underline{x}(t)$, t is the parameter. You know that differentiating this parametrization to t is giving you the velocity $\underline{v}(t)$, a vector tangent to the

curve. Differentiating the velocity gives the acceleration $\underline{a}(t)$ of that particle. The acceleration can be divided in two parts, the part parallel to the velocity vector, this part takes care for altering the velocity, in the sense that the length of the velocity vector changes. From the part of the acceleration vector perpendicular to the velocity the particle is bending.
But particles in space can also rotate, you can have a particle staying at the origin for all the values of t, but rotating.
Looking at fluids, you have to be aware, that just looking at one particle is not enough, a small region of particles can also rotate. A fluid can behave like a syrop. You can look to a velocity field of a fluid or gas. Such a velocityfield depends on several variables. Keep in mind that a movement of such a little region in some fluid, consists out of three movements, 1) just a uniform movement, 2) a straining movement and 3) a rotation.
For example the earth, and in its path around the sun. It is not a good example in the sense that you are not looking at some region of fluid particles. Most of the time the earth is looked as a point mass. The earth rotates following its path, and let's hope no straining movement. It's difficult to find an irrotational body. Or yes, the needle of a compass, you can walk a circle, but the needle is always pointed to the north. Another example is the flow of air around a wing or body. The aircraft flies through the air, which is effectively at rest. Only in a very small region around the wing or body, the flow rotates, but sometimes this can be neglected. May be sound waves are irrotational, but then on a short time scale, such that you have no diffusion. If the velocityfield of the tornado, with the eye, can be described as the gradient of a potential, you have also a irrotational flow.
But be careful, make the difference between a point mass, stiff body and a certain volume in a flow. In a flow, it isn't one particle that determines the behaviour, but a certain volume of particles have a certain kind of behaviour.
Be also careful, what parameters you are looking. For instance a velocity field, when you are integrating and looking at the rotation of a vectorfield, you are doing something with the space variables. The time variable is not really involved on that moment, in the sense that the vectorfield is not changing in time. You have a velocity field per unit of time, and in this course most of the vectorfields are constant in time. So speaking about deformation of a region, that happens in time, you follow a certain region in time.
Taking the flux, you can calculate on different places in the space of the vectorfield, the amount of fluid going through a surface with a certain kind of shape. You can also look at a fixed place in space and look how the flux is changing in time.
Experts will have very much comments to this part of the sumnmary. My intention is to make you clear that you have to be careful with vectorfields.

So some experts don't like the notation, because it is written in cartesian coordinates and in nature you don't have coordinates. They like it to write it coordinate free. Then you need to use tensors, and that is another part of mathematics.
The best is that the first courses in mathematics should be given coordinate free. Using tensors from the beginning, just as you learn your own language. But we all know, learned a language, it is difficult to learn another language, so also with tensors!

12.1 Rotation

When a vectorfield $\underline{v}$ is given to you, and a little rectangle parallel to the xy-plane. With the flux, you looked at the flow through the surface, and you neglected the flow along the surface. Now you are interested in the flow along the boundary of this rectangle.
The rectangle lies around the point $P = (x, y, z)$, with the corners

$$\begin{aligned}
E &= (x - \frac{\triangle x}{2}, y - \frac{\triangle y}{2}, z) \\
F &= (x + \frac{\triangle x}{2}, y - \frac{\triangle y}{2}, z) \\
G &= (x + \frac{\triangle x}{2}, y + \frac{\triangle y}{2}, z) \\
H &= (x - \frac{\triangle x}{2}, y + \frac{\triangle y}{2}, z)
\end{aligned}$$

The 'flow' from point E to F you define with the following integral

$$\text{Flow}_{EF} = \int_{EF} \underline{v} \bullet dL.$$

In this case, you go over a straight line L, with some parametrization L. You see that you take from the velocity only the component along the curve from E to F. Now you neglect the component perpendicular to this curve.
At the point $P = (x, y, z)$, you have a velocity vector $\underline{v} = (v_1, v_2, v_3)$. At the midpoint of the line EF, you have the following velocity in the x-direction

$$\text{vmidp}_{EF} = (v_1 - \frac{\partial v_1}{\partial y}(\frac{\triangle y}{2}))$$

just using the linear approximation of v_1 in the y-direction. At the midpoints of the other sides, you get similar approximations

$$\begin{aligned}
\text{vmidp}_{FG} &= (v_2 + \frac{\partial v_2}{\partial x}(\frac{\triangle x}{2})) \\
\text{vmidp}_{GH} &= (v_1 + \frac{\partial v_1}{\partial y}(\frac{\triangle y}{2})) \\
\text{vmidp}_{HE} &= (v_2 - \frac{\partial v_2}{\partial x}(\frac{\triangle x}{2}))
\end{aligned}$$

You go from point E to F and so on back to point E, along the sides of the rectangle and the 'flow' along this curve, also called the circulation, becomes approximately

$$\begin{gathered}
\text{Flow}_{EFGHE} = \int_{EFGHE} \underline{v} \bullet dL \doteq \\
\text{vmidp}_{EF}\triangle x + \text{vmidp}_{FG}\triangle y - \\
\text{vmidp}_{GH}\triangle x - \text{vmidp}_{HE}\triangle y
\end{gathered}$$

Be careful with the direction of integration! Working out the approximation above gives you

$$\begin{gathered}
\text{Flow}_{EFGHE} = \int_{EFGHE} \underline{v} \bullet dL \doteq \\
(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})\,\triangle x\,\triangle y
\end{gathered}$$

In books you find that the term

$$(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})$$

is the z-component of the rotation of the vectorfield $\underline{v}$. You see also something else that is interesting, if you integrate this term over the rectangle you get the circulation. So you have found a relation between an integral over a curve and an integral over a surface! This is a something, which has to do with the theorems of Stokes and Green.
In rectangles parallel to the yz- and xz-plane you can do the same calculation and find the rotation of the vectorfield $\underline{v}$ in the x and y direction. These terms putting into a vector gives the rotation of the vectorfield $\underline{v}$, notation: curl($\underline{v}$). Another way in calculating the curl($\underline{v}$) is as follows

$$\text{curl}(\underline{v}) = \bigtriangledown \times \underline{v},$$

this means writing out the outer product of $\bigtriangledown$ with $\underline{v}$,

$$\text{curl}(\underline{v}) = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

(In the dutch language, we have the grad, div and rot and in the english language, the grad, div and (rot or curl).)

12.2 Move a little piece of a line

You look at a velocityfield in the 3 dimensional space and this has the following Cartesian coordinates

$$\underline{v} = (v_1, v_2, v_3).$$

and assume that the vectorfield is independent of the time.
Take two points A and B at time t:

$$A := \underline{x},\ B := \underline{x} + d\underline{x}.$$

You wants to know what is happened after a little time dt, the points moved, so at time $t + dt$ the situation is

$$A := \underline{x} + \underline{v}(\underline{x})\, dt,\ B := (\underline{x} + d\underline{x}) + \underline{v}(\underline{x} + d\underline{x})\, dt.$$

Now you can write out the term

$$\underline{v}(\underline{x} + d\underline{x})$$

with the vector $d\underline{x} = (dx, dy, dz)$ and using the gradients on a correct way, you get

$$\underline{v}(\underline{x} + d\underline{x}) = \underline{v}(\underline{x}) + \begin{pmatrix} \frac{d\,v_1}{dx} \\ \frac{d\,v_2}{dx} \\ \frac{d\,v_3}{dx} \end{pmatrix} dx + \begin{pmatrix} \frac{d\,v_1}{dy} \\ \frac{d\,v_2}{dy} \\ \frac{d\,v_3}{dy} \end{pmatrix} dy + \begin{pmatrix} \frac{d\,v_1}{dz} \\ \frac{d\,v_2}{dz} \\ \frac{d\,v_3}{dz} \end{pmatrix} dz.$$

You are interested in what is happened with the little piece $d\underline{x}$. So look at time $t + dt$

$$B - A = d\underline{x} + (\begin{pmatrix} \frac{d\,v_1}{dx} \\ \frac{d\,v_2}{dx} \\ \frac{d\,v_3}{dx} \end{pmatrix} dx + \begin{pmatrix} \frac{d\,v_1}{dy} \\ \frac{d\,v_2}{dy} \\ \frac{d\,v_3}{dy} \end{pmatrix} dy + \begin{pmatrix} \frac{d\,v_1}{dz} \\ \frac{d\,v_2}{dz} \\ \frac{d\,v_3}{dz} \end{pmatrix} dz\,)\, dt.$$

You see that during this timestep, the little $d\underline{x}$ has changed. What happended with the original $d\underline{x}$? But let first look at the notation of all these vectors and write them in a manageable way. They have so-called matrices and in this case, you need the following matrix, called $\nabla\underline{v}$

$$\nabla\underline{v} = \begin{bmatrix} \frac{d\,v_1}{dx} & \frac{d\,v_1}{dy} & \frac{d\,v_1}{dz} \\ \frac{d\,v_2}{dx} & \frac{d\,v_2}{dy} & \frac{d\,v_2}{dz} \\ \frac{d\,v_3}{dx} & \frac{d\,v_3}{dy} & \frac{d\,v_3}{dz} \end{bmatrix}$$

and

$$(\nabla\underline{v})\,d\underline{x} = \begin{pmatrix} \frac{d\,v_1}{dx} \\ \frac{d\,v_2}{dx} \\ \frac{d\,v_3}{dx} \end{pmatrix} dx + \begin{pmatrix} \frac{d\,v_1}{dy} \\ \frac{d\,v_2}{dy} \\ \frac{d\,v_3}{dy} \end{pmatrix} dy + \begin{pmatrix} \frac{d\,v_1}{dz} \\ \frac{d\,v_2}{dz} \\ \frac{d\,v_3}{dz} \end{pmatrix} dz$$

You see the relation between the columns of $\nabla\underline{v}$ and the coordinates of $d\underline{x}$. And you have a little bit handsome formula

$$B - A = d\underline{x} + ((\nabla\,\underline{v})\,d\underline{x})\,dt.$$

With matrices, you can do all kind of manipulations, one of them is to transpose a matrix, this means that you mirror the values of the matrix in its diagonal. When M is the matrix, the transposed matrix is notated bij M^T. You have to give an example, such that somebody else sees what you are doing, so a matrix M and its transposed matrix M^T

$$M = \begin{bmatrix} 1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & 8 & 9 \end{bmatrix},\ M^T = \begin{bmatrix} 1 & 4 & 7 \\ -2 & 5 & 8 \\ -3 & -6 & 9 \end{bmatrix}.$$

Doing such kind of manipulations, you come quick to special kind of matrices. For instance, symmetric matrices, this are matrices with $M = M^T$ and so called skew-symmetric matrices $M = -(M^T)$. Why all this manipulations? Matrices which are symmetric or skew-symmetric have special properties and may be, you can use these properties to understand what the matrix $\nabla\,\underline{v}$ has

done with the little vector $d\underline{x}$ within the timestep dt.
If you are a little bit playing with these symmetric and skew-symmetric matrices then it is easily to see, that a squared matrix can always be written as the sum of a symmetric and skew-symmetric matrix. It looks a little bit on the formulas of the cos and the sin functions, when you express them in the complex exponential function. (The complex conjugate is replace by the transposition and on this moment you don't have to worry about imaginary parts. If the matrix has complex numbers, you have to do other things, but this is out of the scope on this moment.)
Every squared matrix M, you can write as

$$M = \frac{M + M^T}{2} + \frac{M - M^T}{2},$$

try it out with the above given matrix with the numbers in it. You see that symmetric part is

$$\frac{M + M^T}{2}$$

and the skew-symmetric part is

$$\frac{M - M^T}{2}$$

If you try it out on the matrix with numbers, you see that the skew-symmetric matrix has always zeros on the diagonal.
You are busy with vectorfields and the matrix $\nabla\,\underline{v}$, so we rewrite this matrix in a symmetric part and a skew-symmetric part

$$\nabla\,\underline{v} = \frac{\nabla\,\underline{v} + (\nabla\,\underline{v})^T}{2} + \frac{\nabla\,\underline{v} - (\nabla\,\underline{v})^T}{2}.$$

Writing out the skew-symmetric part gives you

$$SR = \frac{\nabla\,\underline{v} - (\nabla\,\underline{v})^T}{2} =$$

$$\begin{bmatrix} 0 & \frac{1}{2}\,(\frac{d\,v_1}{dy} - \frac{d\,v_2}{dx}) & \frac{1}{2}\,(\frac{d\,v_1}{dz} - \frac{d\,v_3}{dx}) \\ -\frac{1}{2}\,(\frac{d\,v_1}{dy} - \frac{d\,v_2}{dx}) & 0 & \frac{1}{2}\,(\frac{d\,v_2}{dz} - \frac{d\,v_3}{dy}) \\ -\frac{1}{2}\,(\frac{d\,v_1}{dz} - \frac{d\,v_3}{dx}) & -\frac{1}{2}\,(\frac{d\,v_2}{dz} - \frac{d\,v_3}{dy}) & 0 \end{bmatrix}.$$

Looking at the coëffcients in the matrix, you see known expressions, you see the components of the rotation vector of the vectorfield $\underline{v}$, the components of this vector are only divided by 2.
The symmetrix part becomes

$$SD = \frac{\nabla \underline{v} + (\nabla \underline{v})^T}{2} =$$

$$\begin{bmatrix} \frac{d v_1}{dx} & \frac{1}{2}(\frac{d v_1}{dy} + \frac{d v_2}{dx}) & \frac{1}{2}(\frac{d v_1}{dz} + \frac{d v_3}{dx}) \\ \frac{1}{2}(\frac{d v_1}{dy} + \frac{d v_2}{dx}) & \frac{d v_2}{dy} & \frac{1}{2}(\frac{d v_2}{dz} + \frac{d v_3}{dy}) \\ \frac{1}{2}(\frac{d v_1}{dz} + \frac{d v_3}{dx}) & \frac{1}{2}(\frac{d v_2}{dz} + \frac{d v_3}{dy}) & \frac{d v_3}{dz} \end{bmatrix}.$$

But now, what do you do with the matrices SR and SD? When you have a symmetric matrix, for instance SD, you can canculate, three vectors which are perpendicular to each other. Taking one of these vectors, and you calculate the product of the matrix with that vetor (direction), you get the same vector back, but the length and/or direction of the vector can be changed. For instance a ball can be changed in a cigar (ellipsoide). (These vectors are called eigenvectors, beloning to certain eigenvalues.)
This means that the matrix SD, the symmetric part of $\nabla \underline{v}$, deforms the little piece $d\underline{x}$. Taking a little square, made out of these little pieces, it can be deformed to a parallellogram, after the timestep dt.
The skew-symmetric part of $\nabla \underline{v}$ is doing something else. Writing out the product of the matrix SR and the little piece $d\underline{x}$, you see that

$$(SR)\, d\underline{x} = \frac{1}{2} \operatorname{rot}(\underline{v}) \times d\underline{x}.$$

(You know, multiply each column of the matrix SR with the correct coordienate of $d\underline{x}$ and sum them together.)
The vector $\frac{1}{2} \operatorname{rot}(\underline{v})$ is the angular velocity of $d\underline{x}$. The matrix SR gives us a stiff rotation.
About a skew-symmetric matrix, it is difficult to give general properties, it depends on the measure of the matrix, an odd or even amount of columns and the dimension is an important fact. In three dimension you could tell something, but this is out of the scope of this lecture. But if you take the $d\underline{x}$ in the direction of the rotation vector, you get the zero vector out of it. And you can look, wat is happening when you take a little surface and and

a rotation vector which is pendicular to that surface.
Conclusion: After a little time step dt, the distance between the moved points A and B has changed, due to a deformation and a stiff rotation.
At time t: $B - A = d\underline{x}$.
At time $t + dt$: $B - A = d\underline{x} + [(SD)\, d\underline{x} + \frac{1}{2}\,\mathrm{rot}(\underline{v}) \times d\underline{x}]$, the deformation and stiff rotation.

13 Theorem of Green, Integral theorem of Gauss

You try to derive these theorems on your own way. With the knowledge of the circulation, you can derive the theorem of Green. With the knowledge of the divergence theorem you derive the integral theorem of Gauss.

13.1 Theorem of Green

The theorem of Green, you have already derived in the preceding section. You have some domain G in the xy-plane with a boundary C and a vectorfield $\underline{F} = (F_1, F_2)$ then the theorem of Green tells you

$$\int\int_G \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dG = \int_C (F_1\, dx + F_2\, dy).$$

You can do as if you are working with something out of the three dimensional space, so make a new vectorfield $\underline{F} = (F_1, F_2, 0)$ and integrate over the domain G, lying in the plane $z = 0$. If you divide your domain in little rectangles and calculate the circulation on each reactangle and add all these parts together, you get the theorem of Green. The function $(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$ is the z-component of the rotation vector of the new defined vectorfield F. Draw two rectangles with one side equal to them, the sum of the line integrals over this side gives zero, because you are integrating in oppossite directions over this side.The line integral becomes an integral over the your domain G.

13.2 Integral theorem of Gauss

There is no easier way then to define a new vectorfield, for instance $\underline{F} = (f(x, y, z), 0, 0)$, f is function, just giving us a scalar, for certain values of

x, y and z. Assume that the vector $\underline{n}$ has the following components $\underline{n} = (n_1, n_2, n_3)$. Use the divergence theorem and you get

$$\int\int\int_{\mathcal{V}} \frac{\partial f}{\partial x}\, d\mathcal{V} = \int\int_{\mathcal{S}} (f\, n_1)\, d\mathcal{S},$$

This is going well, so changing the vectorfield on the good way and you also get

$$\int\int\int_{\mathcal{V}} \frac{\partial f}{\partial y}\, d\mathcal{V} = \int\int_{\mathcal{S}} (f\, n_2)\, d\mathcal{S},$$

$$\int\int\int_{\mathcal{V}} \frac{\partial f}{\partial z}\, d\mathcal{V} = \int\int_{\mathcal{S}} (f\, n_3)\, d\mathcal{S}.$$

14 Theorem of Stokes

The idea is that you can derive the theorem of Stokes, just using the theorem of Green. Using Green you need a domain of integration on for instance the xy-plane, so you need to project your surface, and the boundary of that surface, on the xy-plane and this will give all kind of correction terms. Correction terms on the surface, but also to the line integral on the boundary of that surface. You have to be careful, but it has to succeed.

14.1 Some calculations

What do you have? First a surface $\mathcal{S}$

$$\mathcal{S} : z = f(x, y), \text{ with } (x, y) \in \mathcal{S}^{'}.$$

With $\mathcal{S}^{'}$ is meant the projection of $\mathcal{S}$ on the xy-plane. Here you get the following parametrization of that surface

$$\mathcal{S} : \underline{x}(x, y) = (x, y, f(x, y)) \text{ with } (x, y) \in \mathcal{S}^{'}.$$

The surface $\mathcal{S}$ has a boundary called $\mathcal{C}$, which is also projected on the xy-plane and this projection is called $\mathcal{C}^{'}$. You are careful with all kind of directions of normals on the surface and you take care to the direction of you are walking on the boundary $\mathcal{C}$. You take in mind a surface looking like the upper part of a ball, the normals have always a positive component in the z-direction and along the boundary $\mathcal{C}$, you are always walking in the conter-clockwise direction.
Further you see that when you have a scalar function v,

$$v(x, y, z) \text{ some scalar, with } (x, y, z) \in \mathbb{R}^3,$$

and you want to integrate this function over the surface $\mathcal{S}$, you can also integrate the function w

$$w(x, y, z) \,=\, v(x, y, f(x, y)) \,\text{with}\, (x, y, z) \in \mathbb{R}^3.$$

over that surface $\mathcal{S}$. On that surface $\mathcal{S}$ these scalar functions v and w have the same value.
First some facts, for instance

$$d\mathcal{S} \,=\, |\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}|\, dx\, dy \,=\, \sqrt{1 \,+\, (\frac{\partial f}{\partial x})^2 \,+\, (\frac{\partial f}{\partial y})^2}\, dxdy \,=\, \sqrt{1 \,+\, (\frac{\partial f}{\partial x})^2 \,+\, (\frac{\partial f}{\partial y})^2}\, d\mathcal{S}^{'}.$$

What about the relation between the unit tangent vector $\underline{t}$ at $\mathcal{C}$ and the unit tangent vector $\underline{t}^{'}$ at $\mathcal{C}^{'}$? The projection of $\underline{t}$ makes the z-coordinate equal to zero and is tangent to $\mathcal{C}^{'}$, further no changes. Such that if

$$\underline{t} \,=\, (t_1, t_2, t_3) \;\text{ is an unit vector}$$

then you get

$$\underline{t}^{'} \,=\, \frac{(t_1, t_2, 0)}{\sqrt{t_1^2 \,+\, t_2^2}} \;\text{ also a unit vector}$$

It is also interesting to see that

$$\underline{t} \bullet \underline{t}^{'} \,=\, |\underline{t}|\, |\underline{t}^{'}|\, \cos\phi \,=\, \sqrt{t_1^2 \,+\, t_2^2}$$

and comparing the lenghts of the tangent vectors, you get

$$d\mathcal{C}^{'} \,=\, \cos\phi\, d\mathcal{C} \,=\, \sqrt{t_1^2 \,+\, t_2^2}\, d\mathcal{C}.$$

When you integrate over $\mathcal{C}$ or $d\mathcal{C}^{'}$, it would be nice to have some relation between certain coordinates of the tangent vectors, for instance

$$d\mathcal{C}\, (t_1, t_2, 0) \,=\, d\mathcal{C}\, \sqrt{t_1^2 \,+\, t_2^2}\, (t_1^{'}, t_2^{'}, 0) \,=\, d\mathcal{C}^{'}\, (t_1^{'}, t_2^{'}, 0).$$

Only you have nothing done with the function w, you defined before, giving the same values as the function v on the surface $\mathcal{S}$, you have only to give values of x and y.
Your idea is that with the projection on the xy-plane you can may be use the theorem of Green. But then you have to intergrate over the surface $\mathcal{S}$ some function with the form

$$\frac{\partial}{\partial x}(..) \,-\, \frac{\partial}{\partial y}(..).$$

Do you have the ingredients to make such a function? Try something, so

$$\frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial f}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial f}{\partial y}$$

and playing a little bit gives you

$$\frac{\partial v}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(w\,\frac{\partial f}{\partial y}) - \frac{\partial}{\partial y}(w\,\frac{\partial f}{\partial x}),$$

something with the structure, you searched.
Looking into the formula of Stokes, you see the unit normal on the surface $\mathcal{S}$. To calculate the correction factor you have calculated an outer product which stays perpendicular to the surface $\mathcal{S}$

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y},$$

a vector with a positive z-coordinate. Most of the time they ask an unit normal $\underline{n}$ so,

$$\underline{n} = \frac{\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}}{|\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}|} = \frac{(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)}{\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} = (n_1, n_2, n_3).$$

Let's try.

14.2 Calculating a part of the integral

Seeing the part, which you can calculate

$$\int\int_{\mathcal{S}}(\frac{\partial v}{\partial y}\,n_1 - \frac{\partial v}{\partial x}\,n_2)\,d\mathcal{S} =$$

$$\int\int_{\mathcal{S}'}(\frac{\partial}{\partial x}(w\,\frac{\partial f}{\partial y}) - \frac{\partial}{\partial y}(w\,\frac{\partial f}{\partial x}))\,d\mathcal{S}' =$$

use theorem of Green

$$\int_{\mathcal{C}'} w\,(\frac{\partial f}{\partial x}\,t_1' + \frac{\partial f}{\partial y}\,t_2')\,d\mathcal{C}' =$$

$$\int_{\mathcal{C}} v\,(\frac{\partial f}{\partial x}\,t_1 + \frac{\partial f}{\partial y}\,t_2)\,d\mathcal{C}$$

Should it go wrong or not?
The integrand looks a little bit an inner product. The normal $\underline{n}$ of the surface $\mathcal{S}$ and the tangent vector along $\underline{\mathcal{C}}$ are perpendicular to each other, which means

$$\underline{n} \bullet \underline{t} = 0.$$

Writing out gives you

$$t_3 = \frac{\partial f}{\partial x} t_1 + \frac{\partial f}{\partial y} t_2$$

The result becomes

$$\int \int_{\mathcal{S}} (\frac{\partial v}{\partial y} n_1 - \frac{\partial v}{\partial x} n_2) \, d\mathcal{S} = \int_{\mathcal{C}} v \, t_3 \, d\mathcal{C}$$

Changing the variables x, y and z, $x \rightarrow y \rightarrow z \rightarrow x$
and $n_1 \rightarrow n_2 \rightarrow n_3 \rightarrow n_1$ and $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_1$ gives you other results

$$\int \int_{\mathcal{S}} (\frac{\partial v}{\partial z} n_2 - \frac{\partial v}{\partial y} n_3) \, d\mathcal{S} = \int_{\mathcal{C}} v \, t_1 \, d\mathcal{C}$$

$$\int \int_{\mathcal{S}} (\frac{\partial v}{\partial x} n_3 - \frac{\partial v}{\partial z} n_1) \, d\mathcal{S} = \int_{\mathcal{C}} v \, t_2 \, d\mathcal{C}$$

Nice results but how to use, you want to get the formula of Stokes!

14.3 Theorem of Stokes

You have just derived some results for functions and Stokes is saying something about an integral of a vectorfield $\underline{v} = (v_1, v_2, v_3))$. The idea is take the function v_1 within the result where you have t_1 and so on. Add these results together, it are all integrals over the curve $\mathcal{C}$, so no problem. This gives

$$\int_{\mathcal{C}} (v_1 \, t_1 + v_2 \, t_2 + v_3 \, t_3) \, d\mathcal{C} = \int_{\mathcal{C}} \underline{v} \bullet \underline{t} \, d\mathcal{C}$$

and adding together and some rearrangement

$$\int \int_{\mathcal{S}} (\frac{\partial v_1}{\partial z} n_2 - \frac{\partial v_1}{\partial y} n_3) + (\frac{\partial v_2}{\partial x} n_3 - \frac{\partial v_2}{\partial z} n_1) + (\frac{\partial v_3}{\partial y} n_1 - \frac{\partial v_3}{\partial x} n_2) \, d\mathcal{S} =$$

$$\int \int_{\mathcal{S}} ((\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}) n_1 + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}) n_2 + (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) n_3) \, d\mathcal{S} =$$

$$\int \int_{\mathcal{S}} \underline{n} \bullet \mathrm{curl}(\underline{v}) \, d\mathcal{S}.$$

The theorem of Stokes is a fact:

$$\int_{\mathcal{C}} \underline{v} \bullet \underline{t} \, d\mathcal{C} = \int \int_{\mathcal{S}} \underline{n} \bullet \mathrm{curl}(\underline{v}) \, d\mathcal{S}.$$

15 Examples

The summary is no summary anymore.
Most of the time there will be written i, but in this course it has to be as j.
Also the conjugate of a complex number is written as $\overline{z}$ instead of z^*, sorry.

15.1 Complex number(s)

Given two complex numbers, you can multiply them, divide them, take the argument, the radius, and take the complex conjugate if necessary.
Let's take two complex numbers

$$z_1 = 1 - \sqrt{3}\, i,\; z_2 = \exp{(\sqrt{3} + \frac{\pi}{6} i)}.$$

To add or subtract them, you have to write them out, in a real and imaginary part.

$$z_2 = \exp{(\sqrt{3})}\,(\cos(\frac{\pi}{6}) \,+\, i\, \sin(\frac{\pi}{6})) \,=\, \exp{(\sqrt{3})}\,(\frac{1}{2}\sqrt{3} \,+\, i\,\frac{1}{2})$$

To add or to substract these complex numbers is no problem anymore.
Multiplication or dividing these numbers. The most easily way, is that bove numbers are written in the form radius times exponential function. The number z_2 is given in such a form, the number z_1

$$z_1 \,=\, 2\,(\frac{1}{2} \,-\, i\,\frac{1}{2}\sqrt{3}) \,=\, 2\, \exp{(-i\,\frac{\pi}{3})}$$

For instance dividing z_1 and z_2

$$\frac{z_1}{z_2} \,=\, \frac{2\, \exp{(-i\,\frac{\pi}{3})}}{\exp{(\sqrt{3} + \frac{\pi}{6} i)}} \,=\, \frac{2}{\exp{(\sqrt{3})}}\,\frac{\exp{(-i\,\frac{\pi}{3})}}{\exp{(i\,\frac{\pi}{6})}} \,=\, \frac{2}{\exp{(\sqrt{3})}}\, \exp{(-i\,\frac{\pi}{3} \,-\, i\,\frac{\pi}{6})}$$

You see, what happens, the radii are divided and the arguments of the two numbers are subtracted of each other. The result becomes

$$\frac{z_1}{z_2} \,=\, -i\,\frac{2}{\exp{(\sqrt{3})}}$$

Playing with these numbers

$$\frac{\overline{z_1}}{z_2} \,=\, \frac{\overline{2\, \exp{(-i\,\frac{\pi}{3})}}}{\exp{(\sqrt{3} + \frac{\pi}{6} i)}} \,=\, \frac{2\, \exp{(i\,\frac{\pi}{3})}}{\exp{(\sqrt{3} + \frac{\pi}{6} i)}},$$

writing out gives you

$$\frac{\overline{z_1}}{z_2} \,=\, \frac{2}{\exp{(\sqrt{3})}}\, \exp{(i\,\frac{\pi}{6})} \,=\, \frac{2}{\exp{(\sqrt{3})}}\,(\frac{1}{2}\sqrt{3} \,+\, i\,\frac{1}{2}).$$

The real and imaginary part can be given and the argument with $(+\,k\,2\pi)$!

15.2 Differential equation(s)

You take the inhomogeneus differential equation

$$\frac{d^2\,x}{dt^2}(t) \,+\, 4\,\frac{d\,x}{dt}(t) \,+\, 13\,x(t) \,=\, (\,2 \,+\, t\,)\,\sin(2t).$$

First solve the homogeneous equation, giving you $x_H(t)$ and then search for a particular solution $x_P(t)$.
The homogeneous equation has always solution of the form $\exp(\lambda\,t)$. Putting it in your homogeneous differential equation, you get

$$(\lambda^2 \,+\, 4\,\lambda \,+\, 13\,\lambda)\,\exp(\lambda\,t) \,=\, 0$$

and this for all the values of t, which means that

$$\lambda^2 \,+\, 4\,\lambda \,+\, 13\,\lambda \,=\, 0.$$

Solutions of this equation are

$$\lambda_1 \,=\, -2 \,+\, 3\,i, \lambda_2 \,=\, -2 \,-\, 3\,i \,=\, \overline{\lambda_1}.$$

So two solutions of the homogeneous equation are easily calculated

$$x_1(t) \,=\, \exp(-2\,t)\,\cos(3\,t),\; x_2(t) \,=\, \exp(-2\,t)\,\sin(3\,t),$$

using for instance

$$x_1(t) \,=\, \mathrm{Re}(\exp(\lambda_1\,t)), x_2(t) \,=\, \mathrm{Im}(\exp(\lambda_1\,t))$$

You have found

$$x_H(t) \,=\, A\,x_1(t) \,+\, B\,x_2(t),$$

A and B are real constants.
Difficult is to search the particular solution. But $\sin(2\,t) \,=\, \mathrm{Im}(\exp(i\,2\,t))$.
You solve the differential equation

$$\frac{d^2\,x}{dt^2}(t) \,+\, 4\,\frac{d\,x}{dt}(t) \,+\, 13\,x(t) \,=\, (\,2 \,+\, t\,)\,\exp(i\,2\,t)$$

and afterwards taking the imaginary part of the solution, gives you the particular solution. The solution of the homogeneous equation has nothing in common with inhomogeneous part of the differential equation, this means no extra t's or such things. So you try the following complex solution

$$x_P(t) \,=\, (A \,+\, B\,t)\,\exp(i\,2\,t),$$

A and B are may be complex numbers complex numbers.
The try-out solution is put into the equation and after some rearrangements you get

$$\exp(2\,i\,t)\,(A\,(9\,+\,8\,i)\,+\,B\,((4\,+\,4\,i)\,+\,(9\,+\,8\,i)\,t))\,=\,(\,2\,+\,t\,)\,\exp(i\,2\,t),$$

and you want to get lineair equations for the complex constants A and B. This means, you look for the parts, with a t and without a t, giving

$$A\,(9\,+\,8\,i)\,+\,B\,(4\,+\,4\,i)\,=\,2$$
$$B\,(9\,+\,8\,i)\,=\,1$$

Solving these equations

$$A\,=\,\frac{1966}{21025}\,-\,i\,\frac{1812}{21025}$$

$$B\,=\,\frac{9}{145}\,-\,i\,\frac{8}{145}$$

This are not very nice answers, so when you take something else for the coëfficients 2 and 1 in the inhomogeous term, it gives may be nicer results. But with this A and B you know the complex solution and you have to take the imaginary part of the following function

$$x_P(t)\,=\,((\frac{1966}{21025}\,-\,i\,\frac{1812}{21025})\,+\,(\frac{9}{145}\,-\,i\,\frac{8}{145})\,t)\,\exp(i\,2\,t).$$

You write the complex solution in a real part and imaginary part, write out the exponential function in cos and sin, multiply and rearrange the results. In this case very much work. Doing it well and taking the imaginary part of that expression, it gives for the particular solution of the given differential equation

$$x_P(t)\,=\,-\frac{1812}{21025}\,\cos(2\,t)\,-\,\frac{8}{45}\,t\,\cos(2\,t)\,+\,\frac{1966}{21025}\,\sin(2\,t)\,+\,\frac{9}{45}\,t\,\sin(2\,t).$$

You don't trust the solution, then put it in the differential equation and look what you get for the inhomogeneous term, let's hope $(\,2\,+\,t\,)\,\sin(2t)$.
The total solution of the differential equation becomes:

$$x_T(t)\,=\,x_H(t)\,+\,x_P(t).$$

Most of the time there are also given boundary conditions or initial conditions. Let's do the following initial conditions:

$$x(0)\,=\,\frac{19213}{21025}$$

$$\frac{d\,x}{dt}(0)\,=\,\frac{1748}{189225}$$

Also after a lot of work, you get as result for the unknown constants A and B of the total solution, coming from the part with the solution of the homogeneous equation,

$$\begin{aligned} A &= 1, \\ B &= \frac{2}{3}. \end{aligned}$$

You have to fill in $t = 0$ in the total solution and you have to differentiate the total solution and also to put $t = 0$ in the obtained formula and then you have to solve two linear equations in A and B.

It's a good exercise to make your own second order inhomogeneous differential equation, with for instance boundary conditions. How to do? When you are writing some differential equation on your paper, you don't know the solution. May be it's a idea to start with the solution. What is differential equation belonging to your solution? For instance, take

$$x_T(t) = 4 * \exp(2\,t)\,\cos(t) + 3\,\exp(2\,t)\,\sin(t) + t\,\sin(t)$$

What's are the boundary conditions? You may choose them yourself.
Having done, you try to solve your self-made differential equation. The answer is no problem anymore!

15.3 Linear approximation, tangent plane

You have the scalarfield

$$\phi(x.y.z) = y\,z\,\sin(\pi\,x)$$

and wants to have a linear approximation in the neighbourhood of point $P = (\frac{1}{2}, 1, 1)$.
You know that is almost the same as the formula of Taylor, but with more variables, so

$$\phi(x.y.z) \doteq \phi(\frac{1}{2}, 1, 1) + \mathrm{grad}(\phi)(\frac{1}{2}, 1, 1) \bullet \begin{pmatrix} x - \frac{1}{2} \\ y - 1 \\ z - 1 \end{pmatrix}.$$

You have to calculate

$$\mathrm{grad}(\phi)(\frac{1}{2},1,1) = \begin{pmatrix} \dfrac{d\,\phi}{dx} \\ \\ \dfrac{d\,\phi}{dy} \\ \\ \dfrac{d\,\phi}{dz} \end{pmatrix} (\frac{1}{2},1,1) = \begin{pmatrix} \pi\,x\,y\,z\,\cos(\pi\,x) \\ z\,\sin(\pi\,x) \\ y\,\sin(\pi\,x) \end{pmatrix} (\frac{1}{2},1,1) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The linear approximation at the given point P is

$$\phi(x.y.z) \doteq 1 + (y - 1) + (z - 1).$$

Taking all points in with $\phi(x,y,z) = 1$ then you have a level surface N in the the three dimensional space, all the elements on that surface have the same value as in point P. And you are asking yourself, what should be the equation of the tangent plane in point P, at the level surface N. The orientation of the gradient is always perpendicular to its level surface so the normal of the tangent plane, at point P is

$$\underline{n} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and the linear equation of the tangent plane is

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} x - \dfrac{1}{2} \\ y - 1 \\ z - 1 \end{pmatrix} = 0.$$

Written out $y + z = 2$.
If you want to have a parametrization of this tangent plane, you need two vectors, which are tangent to the level surface, so perpendicular to the normal of the tangent plane, for instance, the vectors $(1.0.0)$ and $(0,1,-1)$. Both vectors have inproduct zero with the normal of the tangent plane at point P.
A parametrization is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

with λ and μ a real scalar. This is one parametrization out of the thousend and one, you can give. Position point can be different, and the two direction vectors can be changed, but be careful with the linear independence of the direction vectors.

15.4 Line integral, potential

Somebody asks you to calculate the following integral

$$\int_K \underline{v}(\underline{x}) \bullet d\underline{x},$$

the vectorfield is given by

$$\underline{v}(x, y, z) = (y^2 + z\,x,\, x^2 + z\,y,\, z)$$

and the curve K is given by the following parametrization

$$\underline{x}(t) = (\cos(t), \sin(t), t),\, 0 \leq t \leq \frac{\pi}{2}.$$

Let's try, you know that you have to use

$$\frac{d\,\underline{x}}{dt}(t) = ((-\sin(t), \cos(t), 1)$$

because

$$\int_K \underline{v}(\underline{x}) \bullet d\underline{x} = \int_K \underline{v}(\underline{x}(t)) \bullet \frac{d\,\underline{x}}{dt}(t)\,dt.$$

The integral becomes

$$\int_K \underline{v}(\underline{x}) \bullet d\underline{x} = \int_0^{\frac{\pi}{2}} \begin{pmatrix} \sin^2(t) + t\cos(t) \\ \cos^2(t) + t\sin(t) \\ t \end{pmatrix} \bullet \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} dt = .$$

$$\int_0^{\frac{\pi}{2}} (-\sin^3(t) + \cos^3(t)) + t)\,dt$$

This can be solved with the use of the formula of Wallis, or use
$\sin^2(t) + \cos^2(t) = 1$ on a clever way.
Integrals having the following structure

$$\int \sin^2(t)\,\cos(t)dt,\, \int \cos^2(t)\,\sin(t)dt,$$

you can calculate by hand.
You think, that is much work ánd time to solve this problem. Another question is, what to do, if you don't see the primitive of the integrand, for instance the integrand is to difficult. Do you know another method, such that it gives you the value of this line integral?

There was something with potentials. So search for a function ϕ, called the potential, such that

$$\underline{v}(x, y, z) = -\text{grad}(\phi)(x, y, z),$$

the vectorfield is written as minus one times the gradient of a function.
Value of the integral becomes the value of the potential at the startpoint of the curve minus the value of the potential at the endpoint of the curve K.
In this case, you can try, ϕ has to satisfy

$$\frac{d\,\phi}{dx} = -(y^2 + z\,x),$$

and you get

$$\phi(x, y, z) = -x\,y^2 - \frac{1}{2} z\,x^2 + h(y, z).$$

Second step

$$\frac{d\,\phi}{dy} = -2\,y\,x + \frac{d\,h}{dy} = -(x^2 + z\,y),$$

the function $h(y, z)$ has to satisfy

$$\frac{d\,h}{dy}(y, z) = 2\,y\,x - (x^2 + z\,y).$$

You see immediately that this is not possible, h is a function of the variables y and z and depends not on the variable x. In the partial derivative of h, you see the variable x, this may not happen, if there should exist a potential ϕ.
That's not nice, the story of subtracting the values of the potential at the begin- and end-point of the curve K from each other, is not going well, in this case, the given vectorfield is not conservative.
You wants a vectorfield with a potential? Take a function $\phi(x, y, z)$ calculate its gradient, putting a minus sign before and you have a vectorfield $\underline{v}$, a so-called conservative vectorfield. Calculate from this vectorfield the potential (answer you know already). You can ask yourself, what happens if you forget the minus sign, is this dramatic, or is it easily to repair?

15.5 Surface integral

A surface with the following description is given

$$\begin{gathered} z = x^2 + y^2 \\ x^2 + y^2 \leq 1. \end{gathered}$$

What form it has? It a part of a parabolic surface. The best is to make a picture of the surface.
If you want to calculate